Supersonic flow past a bluff body with a detached shock
Part I Two-dimensional body

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SUMMARY

The flow at high Mach number past a body with a rounded nose is considered. Viscosity and heat conduction are neglected, and the body is assumed to be two-dimensional and symmetrical about an axis parallel to the incident stream.

An exact solution is first obtained in the case $\gamma \to 1$, $M \to \infty$, where γ is the adiabatic index and M is the Mach number. This solution is then used as the basis of a double expansion in $\delta = (\gamma - 1)/(\gamma + 1)$ and M^{-2}, after the exact solution has been modified to make the series expansion converge near the body. The expansion is carried out as far as the terms of order $(\delta + M^{-2})^2$.

The results are displayed for various values of δ and M^{-2}; typical results are as follows. With $M^{-2} = 0$, $\delta = \frac{1}{6}$ $(\gamma = 1{\cdot}4)$, and for a parabolic body having unit radius of curvature at the nose, the shock is approximately a parabola with radius of curvature 1·822 at the nose. The distance between the body and the shock along the axis of symmetry is 0·323, and the height of the sonic point from this axis is 0·744 both on the shock and on the body. The actual pressure distribution on the body is shown in figure 4, and agrees well with experiment. The pressure falls to zero at a distance 0·86 downstream from the nose of the body, measured along the axis of symmetry. On the assumption that the pressure remains negligible beyond this point, the total drag is 1·39 $\rho_0 V^2$, where ρ_0 is the density and V is the velocity of the incident stream.

INTRODUCTION

It is well-known that a shock wave is produced when a supersonic stream impinges upon a stationary obstacle, and that if the obstacle has a blunt nose the shock wave is curved and lies upstream of the body. It is the purpose of this paper to investigate the flow behind such a curved shock and its position relative to the obstacle producing it. Viscosity and heat conduction are neglected (except, of course, at the shock), and the body is assumed to be two-dimensional and symmetrical about an axis in the direction of the incident stream.

An exact solution is first obtained for the case $\gamma \to 1$, $M \to \infty$, where γ is the adiabatic index and M is the Mach number of the incident stream. The result for the pressure was originally obtained by Busemann (1933; see also Ivey, Klunker & Bowen 1948) using a less direct approach. It has, however, been suggested by Stocker (1955) that this is only a rough approximation to the truth, even for very large Mach numbers, because the pressure field is particularly sensitive to the value of γ.

Since in practice $\gamma - 1$ is fairly small, particularly at the higher temperatures associated with transition through a strong shock wave, it would seem natural to consider the possibility of improving the exact solution by using it as the first term of a series solution in powers of $\gamma - 1$, and in inverse powers of the Mach number. It is, however, not difficult to show that such a solution does not exist, or at least lacks the desirable quality of being uniformly convergent in the neighbourhood of the body. The reason is that in this region the exact solution fails to give the correct approximation (for non-zero $\delta + M^{-2}$) to the magnitude of the velocity, though, as will appear, the direction of the velocity and the pressure are given correctly to a first approximation. The first problem is thus to correct the basic solution. When this is done, an iterative procedure can be used which, in theory, will give the answer to any desired degree of accuracy. In this paper the solution is taken as far as those terms whose magnitude is of the second order of small quantities.

The success of the method is due in part to the use of the von Mises transformation (Goldstein 1938) in which the stream function is used as one of the independent variables. This transformation simplifies the equations of motion, particularly when entropy variations must be taken into account. It was first used in problems of compressible flow by von Kármán & Tsien (1938), and has since been used by several writers. It is particularly suited to the problem considered here.

The shape of the shock, rather than the shape of the body producing it, is assumed to be known *a priori*. The boundary conditions resulting from the transition relations can then be obtained explicitly and applied on a known boundary. In principle it is quite possible, and not more difficult, to solve the problem for a given body shape; but in practice the former approach gives a series which seems to converge more satisfactorily. This is probably due to the fact that, if the shape of the shock is known, all the boundary conditions, and the boundary on which they are applied, are known exactly. When the shape of the body is given, one of the boundary conditions must be applied at the body in the region where the error, even in the improved approximate solution, is relatively large. This will have an effect on the deduced shape of the shock (no longer known exactly), and hence on the boundary conditions applied at the shock in the next approximation.

Working with a known shock imposes no serious limitation on the results. Most of the analysis is in fact carried through assuming a parabolic profile for the shock, and attention is confined to the flow in the important region near the nose. In this region the shape of the body is approximately parabolic, the error being within the accuracy of the present solution.

On the other hand the solution fails to hold far downstream, for it predicts that the pressure on the body eventually becomes zero and changes sign. This is because one of the assumptions made in obtaining the approximate solution is not valid in a region where the pressure, and hence the density, falls to a small fraction of its value at the nose. (Essentially, the reason is that the expansion of ρ^{δ} in powers of δ is not uniformly valid as $\rho \to 0$, and this limits the validity of equation (13).) Experimental evidence does suggest, however, that at high Mach numbers the pressure eventually becomes negligible. Moreover, the conditions at the nose of the body are independent of conditions sufficiently far downstream where supersonic flow is well established. More precisely, the subsonic region at the nose is independent of the shape of the body downstream of the characteristic through the sonic point on the shock. Thus the ultimate breakdown of the solution should not affect its validity near the nose.

Notation and fundamental equations

Let a two-dimensional inviscid uniform stream, with supersonic velocity parallel to the x-axis, impinge upon a stationary blunt nosed body symmetrical about the x-axis. The origin of co-ordinates (x, y) is taken at the vertex of the shock wave which will be formed upstream of the body.

The velocity, pressure and density in the uniform stream are denoted by $(V, 0)$, p_0, ρ_0 respectively. Corresponding quantities in the region of disturbed flow behind the shock are (Vu, Vv), $\rho_0 V^2 p$ and $\rho_0 \rho(\gamma+1)/(\gamma-1)$. The variables u, v, p and ρ so defined are thus non-dimensional. The reason for the factor $(\gamma+1)/(\gamma-1)$ in the representation of the density will appear later. It ensures that ρ remains finite when $\gamma = 1$ and the Mach number of the uniform stream tends to infinity.

The equations of conservation of mass, momentum and entropy are

$$\left.\begin{aligned} \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) &= 0, \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\delta}{\rho}\frac{\partial p}{\partial x}, \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\delta}{\rho}\frac{\partial p}{\partial y}, \\ u\frac{\partial}{\partial x}(p\rho^{-\gamma}) + v\frac{\partial}{\partial y}(p\rho^{-\gamma}) &= 0, \end{aligned}\right\} \tag{1}$$

where δ has been written for $(\gamma-1)/(\gamma+1)$.

The first of equations (1) implies the existence of a stream function ψ such that $\delta\psi_y = \rho u$, $\delta\psi_x = -\rho v$. In the uniform stream, $\psi = y$ and is continuous across the shock wave. Hence, on the shock, $\psi = y$. Along the x-axis and on the body, $\psi = 0$.

If we change to independent variables (ψ, y), equations (1) transform to

$$\left.\begin{aligned}\frac{\partial p}{\partial \psi}-\frac{\partial u}{\partial y} &= 0,\\ u^2+v^2+(1+\delta)p/\rho &= 1+(1-\delta)\delta^{-1}M^{-2},\\ \frac{\partial}{\partial \psi}\left(\frac{u}{v}\right)+\delta\frac{\partial}{\partial y}\left(\frac{1}{\rho v}\right) &= 0,\\ p\rho^{-\gamma} &= f(\psi),\end{aligned}\right\} \tag{2}$$

where f is an arbitrary function of ψ.

It is also suitable to record here the form of equations (1) when z $(=\psi/y)$ and y are used as independent variables. In the (y, z)-plane the body will be $z = 0$, the shock $z = 1$, the axis of symmetry $y = 0$; and we have

$$\left.\begin{aligned}\frac{\partial}{\partial y}(uy) &= \frac{\partial}{\partial z}(p+uz),\\ u^2+v^2+(1+\delta)p/\rho &= 1+(1-\delta)\delta^{-1}M^{-2},\\ \frac{\partial}{\partial y}\left(\frac{\delta y}{\rho v}\right) &= \frac{\partial}{\partial z}\left(\frac{\delta z-\rho u}{\rho v}\right),\\ p\rho^{-\gamma} &= f(yz).\end{aligned}\right\} \tag{3}$$

Furthermore,

$$dx = -\frac{\delta y}{\rho v}dz+\frac{\rho u-\delta z}{\rho v}dy. \tag{4}$$

Since ρv is an odd function of y, it follows that $\dfrac{\partial}{\partial y}\left(\dfrac{\delta y}{\rho v}\right) = 0$ on $y = 0$, and hence, by (3), that $\rho u = \delta z$. Together with the second and fourth of equations (3) (with $f(0)$ determined by the transition relations across a normal shock) this serves to determine all the dependent variables as functions of z along the axis of symmetry.

The distance between the body and the shock along $y = 0$ is, from (4),

$$x_0 = \delta\int_0^1 \lim_{y\to 0}\left(\frac{y}{\rho v}\right)dz. \tag{5}$$

Let the shock be defined by $x = X(y)$ (with $X(0) = 0$), and let $X_y(y)$ denote the derivative of $X(y)$. The transition relations across the shock wave then give the following boundary conditions to be satisfied at $\psi = y$:

$$\left.\begin{aligned}u &= 1-(1-\delta)\left\{\frac{1}{1+X_y^2}-M^{-2}\right\},\\ v &= (1-\delta)X_y\left\{\frac{1}{1+X_y^2}-M^{-2}\right\},\\ p &= (1-\delta)\left\{\frac{1}{1+X_y^2}-\frac{\delta M^{-2}}{1+\delta}\right\},\\ \rho &= \frac{1}{1+(1-\delta)\delta^{-1}M^{-2}(1+X_y^2)}.\end{aligned}\right\} \tag{6}$$

It follows immediately that

$$f(\psi) = (1-\delta)\left\{\frac{1}{1+X_\psi^2(\psi)} - \frac{\delta M^{-2}}{1+\delta}\right\}\left\{1+(1-\delta)\delta^{-1}M^{-2}(1+X_\psi^2(\psi))\right\}^{(1+\delta)/(1-\delta)} \tag{7}$$

From the third equations of (2), we now have

$$\begin{aligned}\frac{u}{v} &= \delta\int_\psi^y \frac{\partial}{\partial y}\left(\frac{1}{\rho v}\right)d\psi + \left(\frac{u}{v}\right)_{\psi=y} \\ &= \delta\frac{\partial}{\partial y}\int_\psi^y\left(\frac{1}{\rho v}\right)d\psi + \left(\frac{u}{v} - \frac{\delta}{\rho v}\right)_{\psi=y}\end{aligned}$$

With the help of equations (6), this gives

$$\left.\begin{aligned}\frac{u}{v} &= X_y(y) + \delta\frac{\partial}{\partial y}\int_\psi^y \frac{1}{\rho v}\,d\psi \\ &= X_y(y) + \delta\int_z^1 \frac{\partial}{\partial y}\left(\frac{y}{\rho v}\right)dz + \frac{\delta z}{\rho v}.\end{aligned}\right\} \tag{8}$$

It follows that the equation of the body is

$$x = X(y) + \delta\int_0^y \frac{1}{\rho v}\,d\psi. \tag{9}$$

Note that, as $y \to 0$, this becomes

$$\delta\int_0^1 \lim_{y\to 0}\left(\frac{y}{\rho v}\right)dz,$$

which by (5), is just the distance between the body and the shock.

The basic solution

We consider first the case $\delta \to 0$, $M^{-2} \to 0$ (such that $\delta^{-1}M^{-2}$ remains finite). Equation (7) then becomes

$$f(\psi) = \frac{1}{1+X_\psi^2(\psi)} + \delta^{-1}M^{-2};$$

and, when this is combined with equations (2), we arrive at the following relations:

$$\frac{\partial u}{\partial y} = \frac{\partial p}{\partial \psi}, \quad u^2+v^2 = \frac{X_\psi^2(\psi)}{1+X_\psi^2(\psi)}, \quad \frac{\partial}{\partial \psi}\left(\frac{u}{v}\right) = 0. \tag{10}$$

Hence $u/v = X_y(y)$, which also follows immediately from (8). The shock wave thus coincides with the body in the physical plane (though not in the (ψ, y)-plane). There is, however, a discontinuity in the rest of the flow variables between the shock and the body, in addition to the discontinuity across the shock. In the (ψ, y)-plane, the solutions of the first two of equations (10) give

$$\left.\begin{aligned}u = \frac{X_\psi(\psi)X_y(y)}{\{1+X_\psi^2(\psi)\}^{1/2}\{1+X_y^2(y)\}^{1/2}}, \quad v = \frac{X_\psi(\psi)}{\{1+X_\psi^2(\psi)\}^{1/2}\{1+X_y^2(y)\}^{1/2}}, \\ p = \frac{1}{1+X_y^2(y)} - \frac{d}{dy}\left\{\frac{X_y}{(1+X_y^2)^{1/2}}\right\}\int_\psi^y \frac{X_\psi(\psi)\,d\psi}{\{1+X_\psi^2(\psi)\}^{1/2}}.\end{aligned}\right\} \tag{11}$$

In particular, on the body ($\psi = 0$), we have

$$p = \sin^2\beta + \sin\beta \frac{d\beta}{dy}\int_0^y \cos\beta\, dy$$

where β is the inclination of the tangent to the horizontal. This is the result originally obtained by Busemann (1933).

Henceforth the shock is assumed to be a parabola with unit radius of curvature at the vertex, so that its equation is $x = \frac{1}{2}y^2$, and equations (11) give

$$\left.\begin{aligned} u &= \frac{\psi y}{(1+\psi^2)^{1/2}(1+y^2)^{1/2}}, \quad v = \frac{\psi}{(1+\psi^2)^{1/2}(1+y^2)^{1/2}},\\ p &= \frac{(1+\psi^2)^{1/2}}{(1+y^2)^{3/2}}, \quad \rho = \frac{(1+\psi^2)^{3/2}}{(1+y^2)^{3/2}\{1+\delta^{-1}M^{-2}(1+\psi^2)\}}. \end{aligned}\right\} \tag{12}$$

In order to improve this approximation, the straightforward approach would be to substitute this basic solution in the neglected terms of (2) and solve the new set of equations as far as the terms of order δ and M^{-2}. Unfortunately this process does not converge near $\psi = 0$. For example, as $\psi \to 0$ $v \sim \psi$ according to (11); and hence the third of equations (2) would give $u/v \sim \delta \log \psi$. The reason is that the solution is not expansible as a power series in δ and M^{-2} when $\psi = 0$. It will appear that the velocity components are in fact $O(\delta + M^{-2})^{1/2}$ on the body. Thus the basic solution (12), which gives $u = v = 0$, is not a valid approximation to u and v for $\psi = 0$, in the sense that the neglected terms are small compared with the term retained. On the other hand, p, ρ and $u/(vy)$ are all finite and non-zero on the body according to (12); and hence for these quantities the basic solution is presumably a valid approximation in the above sense, for the error is presumably $o(1)$ when $(\delta + M^{-2})$ is small. This will in fact be verified when the higher order terms are obtained.

Equations (12) are accordingly used as the basic solution for p, ρ and $u/(vy)$. It is then a simple matter to improve the corresponding solutions for the velocity components.

From equations (2), (7) and (12) we have, *correct to the first order*,

$$\begin{aligned} \frac{p}{\rho} = f(\psi)\,\rho^{\gamma-1} &= f(\psi)\{1+2\delta\log\rho\}\\ &= \frac{1}{(1+\psi^2)}\bigg[1+\delta^{-1}M^{-2}(1+\psi^2)-\delta-2M^{-2}(1+\psi^2)+\\ &\qquad\qquad +3\{\delta+M^{-2}(1+\psi^2)\}\log\left(\frac{1+\psi^2}{1+y^2}\right)\bigg], \end{aligned} \tag{13}$$

so that

$$u^2+v^2 = \frac{1}{1+\psi^2}\left[\psi^2+3\{\delta+M^{-2}(1+\psi^2)\}\log\left(\frac{1+y^2}{1+\psi^2}\right)\right]. \tag{14}$$

When this is combined with $u = yv$, and all but the leading terms are discarded, the following uniformly valid approximations are obtained:

$$\left.\begin{aligned} u &= \frac{y}{(1+y^2)^{1/2}(1+\psi^2)^{1/2}}\left\{\psi^2+3d\log(1+y^2)\right\}^{1/2}, \\ v &= \frac{1}{(1+y^2)^{1/2}(1+\psi^2)^{1/2}}\left\{\psi^2+3d\log(1+y^2)\right\}^{1/2}, \end{aligned}\right\} \tag{15}$$

where d has been written for $(\delta+M^{-2})$.

First-order approximation

From equations (12) and (15), we now have

$$\frac{\delta}{\rho v} = \frac{(d+M^{-2}\psi^2)(1+y^2)^2}{\{\psi^2+3d\log(1+y^2)\}^{1/2}(1+\psi^2)}; \tag{16}$$

and this is uniformly correct as far as terms which are $O(d)$. This is substituted in the right-hand side of the equation

$$\frac{\partial}{\partial\psi}\left(\frac{u}{v}\right) = -\frac{\partial}{\partial y}\left(\frac{\delta}{\rho v}\right),$$

which is then integrated to obtain the next approximation for u/v. The boundary condition is, from (6) with $X_y = y$,

$$\frac{u}{v} = y+\frac{\delta}{y}(1+y^2)+\frac{M^{-2}}{y}(1+y^2)^2 \tag{17}$$

The details are straightforward and are omitted; the only care required is in ensuring that the terms which are discarded are all $o(d)$.

The result is

$$\frac{u}{v} = y+\frac{d\psi y(1+y^2)}{\{\psi^2+3d\log(1+y^2)\}^{1/2}\log(1+y^2)}-y(1+y^2)g, \tag{18}$$

where

$$g = 4d\sinh^{-1}\left[\frac{\psi}{\{3d\log(1+y^2)\}^{1/2}}\right]-2d\log\left[\frac{4y^2}{3d\log(1+y^2)}\right]+ \\ +\frac{d}{y^2}\left[\frac{y^2}{\log(1+y^2)}-1\right]+2(d-M^{-2})\log\left[\frac{1+y^2}{1+\psi^2}\right]-M^{-2}. \tag{19}$$

Before the velocity components are calculated, it is necessary to remove the term in u/v which is singular at $y = 0$. We have already seen that, near $y = 0$, $\rho u \sim \delta z = \delta\psi/y$, or

$$u = U+\frac{d\psi y(1+y^2)^{1/2}}{(1+\psi^2)^{1/2}\log(1+y^2)} \tag{20}$$

say, where U is $O(y^2)$. Equations (15), (18) and (20) then combine to give

$$\frac{U}{v} = y-y(1+y^2)g; \tag{21}$$

and the singular term which appeared in equation (18) has now disappeared. These relations can now be substituted in equation (14) to give

$$U = \frac{-d\psi y^3}{(1+y^2)^{1/2}(1+\psi^2)^{1/2}\log(1+y^2)}+ \\ +\frac{y(1-g)}{(1+y^2)^{1/2}(1+\psi^2)^{1/2}}\left[\psi^2+3(d+M^{-2}\psi^2)\log\left(\frac{1+y^2}{1+\psi^2}\right)\right]^{1/2}; \tag{22}$$

and hence, from (20),

$$u = \frac{d\psi y}{(1+y^2)^{1/2}(1+\psi^2)^{1/2}\log(1+y^2)} + \\ + \frac{y(1-g)}{(1+y^2)^{1/2}(1+\psi^2)^{1/2}}\left[\psi^2+3(d+M^{-2}\psi^2)\log\left(\frac{1+y^2}{1+\psi^2}\right)\right]^{1/2}, \tag{23}$$

$$v = \frac{-d\psi y^2}{(1+y^2)^{1/2}(1+\psi^2)^{1/2}\log(1+y^2)} + \\ + \frac{(1+gy^2)}{(1+y^2)^{1/2}(1+\psi^2)^{1/2}}\left[\psi^2+3(d+M^{-2}\psi^2)\log\left(\frac{1+y^2}{1+\psi^2}\right)\right]^{1/2}. \tag{24}$$

The quantities of prime physical interest are now easily obtained. The shape of the body is calculated from (9) using (16). The equation is referred to an origin at the vertex of the shock, and so automatically gives the distance between the body and the shock. The equation is, correct to the first order,

$$x = \tfrac{1}{2}y^2+\tfrac{1}{2}d(1+y^2)^2\log\left\{\frac{4y^2}{3d(1+y^2)\log(1+y^2)}\right\} + \\ + \tfrac{1}{2}M^{-2}(1+y^2)^2\log(1+y^2). \tag{25}$$

Finally, the pressure is calculated from the equation

$$\frac{\partial p}{\partial \psi} = \frac{\partial u}{\partial y},$$

the right-hand side being given by (23). When this is integrated and combined with the boundary condition $p = (1-\delta)/(1+y^2)$ for $\psi = y$, one finds that

$$\begin{aligned}(1+y^2)^{3/2}p-(1+\psi^2)^{1/2} \\ &= d[\tfrac{3}{2}\log(1+y^2)+3y^2+4-4(1+\psi^2)^{1/2}]\sinh^{-1}\left[\frac{\psi}{\{3d\log(1+y^2)\}^{1/2}}\right] - \\ &- d[\tfrac{3}{4}\log(1+y^2)+\tfrac{3}{2}y^2+2]\log\left[\frac{4y^2}{3d\log(1+y^2)}\left\{\frac{1+(1+\psi^2)^{1/2}}{1+(1+y^2)^{1/2}}\right\}^2\right] + \\ &+ d(1+\psi^2)^{1/2}\left[2-y^{-2}+2\log\left\{\frac{4y^2(1+\psi^2)}{3d(1+y^2)\log(1+y^2)}\right\}\right] + \\ &+ d(1+y^2)^{1/2}(y^{-2}-3)-\tfrac{1}{4}[\{\psi^2+3d\log(1+y^2)\}^{1/2}-\psi]^2 + \\ &+ \tfrac{3}{2}d\int_{\frac{1}{2}\log(1+\psi^2)}^{\frac{1}{2}\log(1+y^2)}\frac{\xi}{\sinh\xi}\,d\xi + \\ &+ M^{-2}(1+\psi^2)^{1/2}\left[8+7y^2+\tfrac{7}{2}\log\left(\frac{1+y^2}{1+\psi^2}\right)\right]-7M^{-2}(1+y^2)^{3/2}. \end{aligned} \tag{26}$$

SECOND-ORDER APPROXIMATION

In principle the approximation can now be carried as far as the second order terms by repeating the process with the first-order approximation replacing the basic solution.

Some care is required in the calculation of the integrand that occurs in equation (9) for the body contour. An expression for v is required which is correct as far as terms which are $O(d^{3/2})$ on the body. Equation (24) is not accurate to this order, since terms have been omitted from within the square brackets which are $O(d^2)$, but which do not tend to zero as $\psi \to 0$. On the body, their contribution to v is $O(d^{3/2})$. Essentially, this means that the first-order solution must be corrected in the same way that the zero-order solution is corrected before proceeding to higher order terms. Thus in the second of equations (2) we use the first-order solution for ρ to get an improved expression for u^2+v^2, analogous to equation (14). This is then combined with the first order solution for u/v to obtain improved expressions for the velocity components.

In practice the algebra becomes prohibitive once the stage of calculating the pressure is reached. Since this investigation is mainly concerned with conditions in the neighbourhood of the nose of the body (which are independent of the behaviour far downstream), the calculation is accordingly restricted to the expansion of the second-order terms omitting fourth and higher powers of y; for this purpose equations (3) have been found superior to equations (2). Actually, the complete expression for the second-order terms is obtained in the case of the equation of the body contour, and is quoted in the appendix; in equation (27) below all fourth and higher powers of y are neglected. Calculations suggest there is little loss of accuracy in ignoring the higher powers of y at least as far as the sonic point on the body. The final results only are quoted below.

The equation of the body is

$$x = \tfrac{1}{2}d\log\frac{4}{3d} + d^2\left(\frac{1}{4}\log\frac{4}{3d} - \frac{3}{8}\right) - \tfrac{1}{2}dM^{-2}\log\frac{4}{3d} + \tfrac{1}{2}M^{-4}\left(\log\frac{4}{3d} - 1\right) +$$
$$+y^2\left[\tfrac{1}{2} + d\log\frac{4}{3d} - \tfrac{1}{4}d + \tfrac{1}{2}M^{-2} + d^2\left(\frac{17}{16}\log^2\frac{4}{3d} + \frac{7}{16}\log\frac{4}{3d} - \frac{161}{96}\right) -\right.$$
$$\left. - dM^{-2}\left(\frac{1}{2}\log\frac{4}{3d} - \frac{1}{6}\right) + M^{-4}\left(\log\frac{4}{3d} - \frac{3}{4}\right)\right] + O(y^4). \quad (27)$$

In the following expression for the pressure on the body ($\psi = 0$), the first-order terms are quoted exactly; in the second-order terms fourth and higher powers of y are omitted. Thus,

$$(1+y^2)^{3/2}p - 1 = d\left[\frac{3}{2}\int_0^{\frac{1}{2}\log(1+y^2)} \frac{\xi}{\sinh\xi}\,d\xi + \{(1+y^2)^{1/2}-1\}\{y^{-2}-3\} - 1 -\right.$$
$$- \{\tfrac{3}{4}\log(1+y^2) + \tfrac{3}{2}y^2\}\log\left\{\frac{4y^2}{3d\log(1+y^2)}\right\} - \tfrac{11}{4}\log(1+y^2) +$$
$$\left. + 2\{\tfrac{3}{4}\log(1+y^2) + \tfrac{3}{2}y^2 + 2\}\log\tfrac{1}{2}\{1+(1+y^2)^{1/2}\}\right] +$$
$$+ M^{-2}[8 + 7y^2 + \tfrac{7}{2}\log(1+y^2) - 7(1+y^2)^{3/2}] - \tfrac{3}{2}M^{-2}(d - M^{-2}) + \tfrac{1}{8}d^2 -$$
$$- 3y^2\left[d^2\left\{\frac{21}{16}\log^2\frac{4}{3d} + \frac{15}{16}\log\frac{4}{3d} + \frac{43}{96}\right\} +\right.$$
$$\left. + dM^{-2}\left\{\frac{3}{4}\log\frac{4}{3d} + \frac{7}{24}\right\} + \frac{3}{4}M^{-4}\left\{\log\frac{4}{3d} - \tfrac{1}{2}\right\}\right]. \quad (28)$$

Numerical calculations

The results embodied in figures 1 to 4 are all based on equations (27) and (28). It was found that the equation of the body was given with sufficient accuracy by equation (27). For example, the neglect of the higher-order

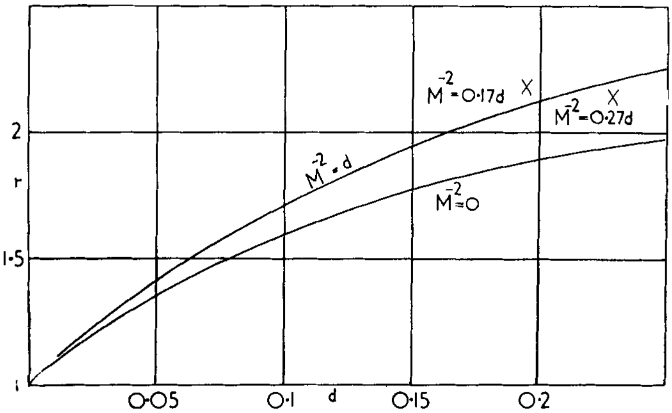

Figure 1. The radius of curvature of the shock on the axis of symmetry for various values of d $(=(\gamma-1)/(\gamma+1)+M^{-2})$. The unit of length is the radius of curvature of the body on the axis of symmetry. The isolated points are experimental values.

terms in y^2 produces an error in the ordinate of the sonic point of only 1% for $\gamma = 1{\cdot}4$, $M^{-2} = 0$. On the other hand, the pressure was calculated from equation (28) as it stands.

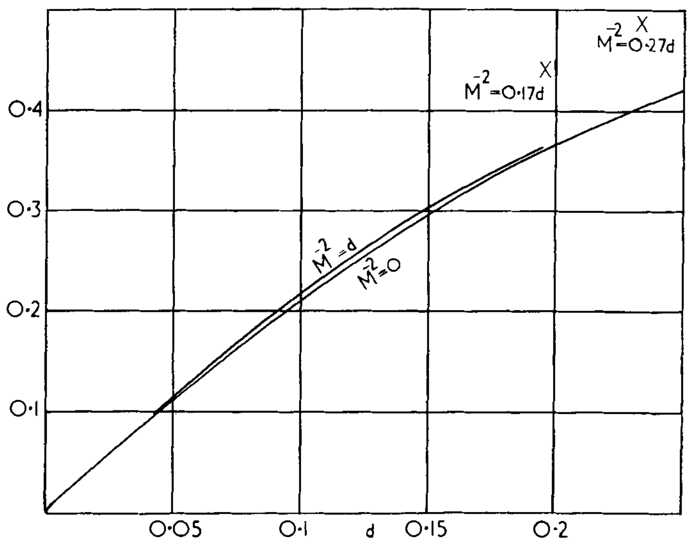

Figure 2. The distance between the body and the shock along the axis of symmetry for various values of d $(=(\gamma-1)/(\gamma+1)+M^{-2})$. The unit of length is the radius of curvature of the body on the axis of symmetry. The isolated points are experimental values.

The previous analysis is based on the assumption of a parabolic shock with unit radius of curvature at the vertex. In practice the results are of more significance when displayed for a given body, and in figures 1 to 4 the results have been adjusted so as to apply to a *body* which is a parabolic cylinder with unit radius of curvature at the vertex.

The radius of curvature of the shock, when $y = 0$ is given in figure 1. If this is denoted by r, the equation of the shock would then be $y^2 = 2rx$. Since the adiabatic index decreases at the high temperatures associated with the conditions behind a strong shock wave, it was thought to be of interest to

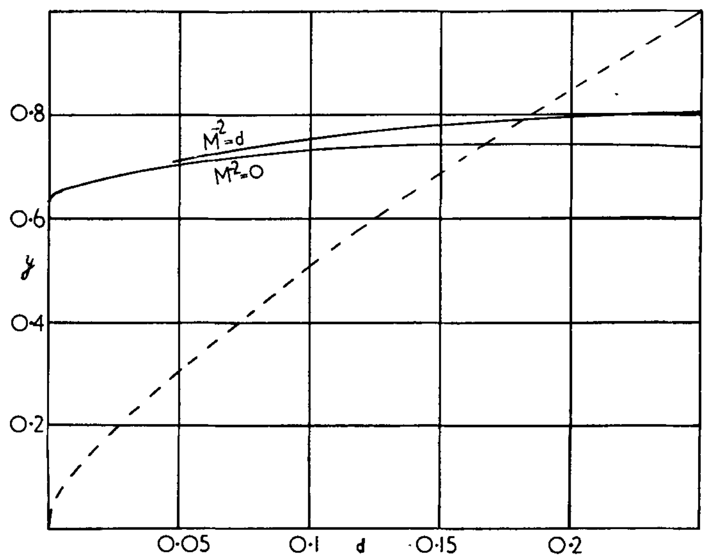

Figure 3. The ordinate of the sonic point ——— on the body - - - - on the shock. When $M^{-2} = d$, the sonic point on the shock is at the vertex.

exhibit the results for various values of δ. On the other hand, since the more significant parameter is d ($= \delta + M^{-2}$), and since $0 < M^{-2} < d$, the variation with d is shown only for the two extreme values of M^{-2}.

Figure 2 shows the distance between the body and the shock, along the line of symmetry, as a fraction of the body radius of curvature. Again the variation is plotted against d for the two extreme values of M^{-2}.

The ordinate of the sonic point is shown in figure 3 for both the body and the shock. On the body the pressure at the sonic point is given by

$$p_s = (1+\delta)^{-1}(1-\delta)(1+\delta-\delta M^{-2})^{-(1-\delta)/2\delta} \qquad (29)$$

This result can be deduced from the second of equations (2). Equations (28) and (29) are together used to obtain figure 3. The ordinate of the sonic point on the shock follows at once from (6) with $X_y = y/r$. Explicitly it is

$$r\left[\frac{\delta(1-M^{-2})}{1-\delta M^{-2}}\right]^{1/2}.$$

When $M^{-2} = d$ (or $\delta = 0$), the sonic point is at the vertex of the shock. It is clear from this fact and from figure 3 that the position of the sonic point on the shock (for a given body shape) is far more sensitive to variations in both d and M^{-2} than the position on the body. When $M^{-2} = 0$, for example, the variation on the body is less than 20% of its minimum value over the range considered in figure 3. In this connection it is interesting to study figure 4, which shows the pressure distribution along the body when $M^{-2} = 0$, and $\gamma = 1$ or 1·4. The former is the exact solution $(1+y^2)^{-3/2}$; the latter is calculated from (28). Figure 4 shows that the pressure in the neighbourhood of the sonic point is even less sensitive to variations in d than the position of

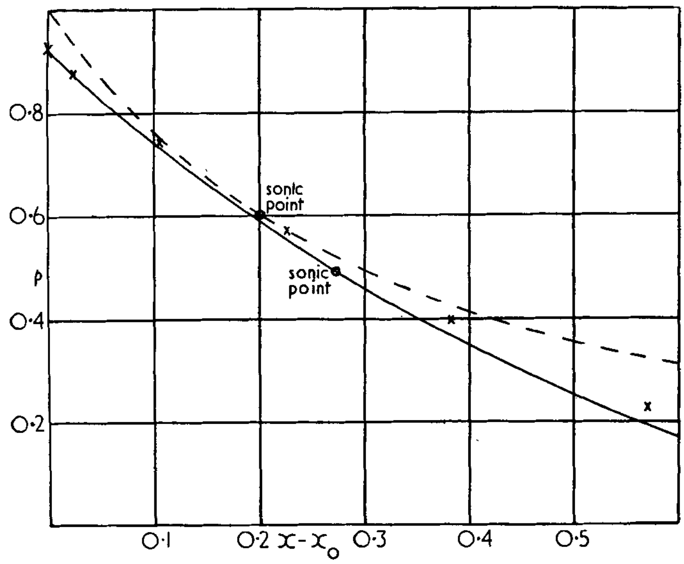

Figure 4. The pressure distribution on the body; the abscissa is the distance along the axis of symmetry, measured from the stagnation point, as a fraction of the body radius of curvature. ——— $M^{-2} = 0$, $d = 1/6$ ($\gamma = 1·4$). - - - - - $M^{-2} = 0$, $d = 0$ ($\gamma = 1$). The crosses denote experimental values for a circular cylinder with $M = 5·8$.

the sonic point itself. Indeed, the Busemann solution would seem to be rather better than has previously been suggested. Moreover the maximum error between the stagnation point and the sonic point (and for some distance beyond) occurs at the stagnation point, where the pressure is known exactly from (7) and the second of equations (2). Explicitly it is given by

$$p_{st} = (1+\delta)^{-1}\{1-\delta^2-\delta(1-\delta)M^{-2}\}^{-\frac{1}{2}(1-\delta)/\delta}. \qquad (30)$$

The experimental results displayed in the figures are taken from a paper by Oliver (1956). In addition, the results of an experiment by Holder (private communication) are shown in figures 1 and 2. Both experiments were carried out on circular cylinders at Mach numbers of 5·8 (Oliver) and 4 (Holder). A value of 1·4 for the adiabatic index has been assumed in plotting these results.

The theory tends to underestimate the stand-off distance and the radius of the shock, judging by the available experimental evidence. The pressure distribution on the body near the nose, however, agrees well with the measurements made by Oliver on a circular cylinder.

When $d = M^{-2} = 0$, the pressure on the body is always positive and approaches zero asymptotically at large distances downstream. For non-zero values of the parameters, however, the pressure becomes zero at a finite distance downstream; and this is borne out by experiment. On the assumption that beyond this point the pressure remains negligible, the total drag on the body would then be (including the upper and lower halves)

$$2\rho_0 V^2 \int p\,dy,$$

the integral being taken from the stagnation point to the point of zero pressure.

When $d = M^{-2} = 0$ the total drag is just $2\rho_0 V^2$ for a parabola with unit radius of curvature at the nose. The corresponding result for $d^{-1} = 6$, $M^{-2} = 0$ is $1{\cdot}39\,\rho_0 V^2$. The pressure becomes zero when $y = 1{\cdot}31$, or $x - x_0 = 0{\cdot}86$ (here $x - x_0$ is the distance downstream measured along the axis of symmetry from an origin at the nose of the *body*).

Appendix

The equation of the body contour, with the second order terms given exactly, is as follows:

$$\begin{aligned} x = {} & \tfrac{1}{2}y^2 + \tfrac{1}{2}dP^4(Q-R) + \tfrac{1}{2}M^{-2}P^4R + \tfrac{1}{8}d^2P^4Q^2(\tfrac{3}{2}R + 7y^2) + \tfrac{1}{2}M^{-4}P^4\{Q - 1 - \\ & - (1-y^2)R^2 - (7+6y^2)R + 14P^2(P-1)\} + \tfrac{1}{4}d^2P^4Q\Big[(1-4y^2)R + 6P - \\ & \quad - 2 - 2(P+1)^{-1} - (3R+6y^2+8)S - 2y^2R^{-1} - 3\int_0^{\frac{1}{2}R} \xi \operatorname{cosech}\xi\, d\xi\Big] + \\ & + \tfrac{1}{4}d^2P^4\Big[(1+2y^2)R^2 - \tfrac{8}{3} + 4\{1 - 3y^2 - y^2R^{-1} - \tfrac{2}{3}(P-1)R^{-1}\}(P+1)^{-1} - \\ & - \tfrac{1}{2}(4y^{-2} - 5)R + 4y^{-2}(1-3y^2)PS + (S^2 + 2S - R + \tfrac{2}{3}R^{-1}S)(3R + 6y^2 + 8) + \\ & \quad + \int_0^{\frac{1}{2}R} \{6 + 2R^{-1} + 3\log(\tfrac{1}{4}y^2 \coth\tfrac{1}{2}\xi)\}\xi \operatorname{cosech}\xi\, d\xi\Big] - \\ & - \tfrac{1}{4}dM^{-2}P^4\Big[\{(7-4y^2)R + 16 + 12y^2 - 14P^3\}Q - (2y^{-2} + 24 + 18y^2)R - \\ & - 4(1-y^2)R^2 + 2(3R + 6y^2 + 8 + 14P^3)S + 4(1 + 4y^2 + 7y^4)(P+1)^{-1} - \\ & \quad - \tfrac{7}{3}R^{-1}(P^2 - P^3 + \tfrac{1}{2}R) - 6\int_0^{\frac{1}{2}R} (2\exp\xi - 1)\xi \operatorname{cosech}\xi\, d\xi\Big], \end{aligned}$$

where

$$P = (1+y^2)^{1/2}, \quad Q = \log\left\{\frac{4y^2}{3d\log(1+y^2)}\right\}$$

$$R = \log(1+y^2), \quad S = \log\tfrac{1}{2}\{1 + (1+y^2)^{1/2}\}.$$

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